

# HOPF QUIVERS

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## Abstract

We classify graded Hopf algebras structures over path coalgebras, that is over free pointed coalgebras, using Hopf quivers which are analogous to Cayley graphs.

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## 1 Introduction

In this paper we provide the classification of path coalgebras which admit a graded Hopf algebra structure; we construct pointed free Hopf algebras in an exhaustive way. Paths of a quiver provide a natural basis of such algebras and the comultiplication of a path is the sum of all the splits of the path. We show that the multiplication of two paths is described through *thin splits* of the paths, a group structure on the vertices and an action of this group on the arrows by permutations on the left and permutations and linear action on the right.

In a previous paper [5] we have presented parts of these results in a dual version since we considered associative path algebras provided by finite quivers. In [5], we have described basic Hopf algebras (simple modules are one-dimensional) instead of pointed Hopf algebras. However this approach appears to be somehow unusual. Note that Radford [13] describes the structure of finite dimensional simple-pointed Hopf algebras generated by pairs of a group-like  $a$  and a  $(1, a)$ -primitive, when the field is algebraically closed.

In the present paper we obtain a complete and more suitable version of the classification which allows infinite quivers. Moreover the description involves formulas for the product besides the canonical formulas for the coproduct. This makes explicit the quantum shuffle product [14] making use of natural elements

in a Hopf bimodule having a simple "geometrical" interpretation in the quiver sense, rather than working systematically with right or left coinvariants.

The first author lectured at MSRI [4] on the present approach in November 1999. The books of C. Kassel [10], S. Montgomery [11], and M.E. Sweedler [15] provide the necessary background for the Hopf algebra side of this paper, while the book of M. Auslander, I. Reiten and S. Smalø [2] gives precise insight to the path algebra approach.

## 2 Path Coalgebras

A quiver  $Q$  is an oriented graph given by two sets  $Q_0$  and  $Q_1$  of vertices and arrows and two maps  $s, t : Q_1 \rightarrow Q_0$  providing each arrow with its source and terminal vertices. Infinite sets  $Q_0$  and  $Q_1$  are allowed. A path  $\alpha$  is a finite sequence of concatenated arrows  $\alpha = a_n \cdots a_1$  which means that  $t(a_i) = s(a_{i+1})$  for  $i = 1, \dots, n-1$ . We set  $s(\alpha) = s(a_1)$  and  $t(\alpha) = t(a_n)$ . Moreover, a vertex  $u$  coincides with its source and terminal vertices. The length of a path is the length of its arrow sequence; vertices are zero-length paths.

If one prefers, a quiver is the same structure than a free category with a given set of free generators :  $Q_0$  is the set of objects, the set of paths is the set of morphisms and  $Q_1$  is the free set of generators. A functor  $F$  from this category to another one is completely determined by the image of the objects and a coherent choice of images of the arrows, meaning that for each arrow  $a$  the morphism  $F(a)$  is from  $F(s(a))$  to  $F(t(a))$ .

Let now  $k$  be a field.

**Definition 2.1** *The coalgebra of paths of a quiver  $Q$  is the linearisation  $kQ$  of the set of paths, equipped with a comultiplication  $\Delta$  and a counit  $\varepsilon$  as follows. For a path  $\alpha = a_n \cdots a_1$  we set*

$$\begin{aligned} \Delta\alpha &= \sum_{i=0}^n a_n \cdots a_{i+1} \otimes a_i \cdots a_1 \\ &= \alpha \otimes s(\alpha) + \sum_{i=1}^{n-1} a_n \cdots a_{i+1} \otimes a_i \cdots a_1 + t(\alpha) \otimes \alpha \\ &= \sum_{\alpha=\alpha_{(2)}\alpha_{(1)}} \alpha_{(2)} \otimes \alpha_{(1)} \end{aligned}$$

where  $\{\alpha_{(2)}\alpha_{(1)}\}$  is the set of splits of  $\alpha$ . Moreover  $\varepsilon(\alpha) = 0$  if  $\alpha$  is a positive length path and  $\varepsilon(u) = 1$  if  $u$  is a vertex.

There is no difficulty to verify that  $kQ$  is indeed a coalgebra. Actually this coalgebra structure is precisely the dual version of the well known path algebra

structure on finite quivers as it is widely used. It has been considered previously by W. Chin and S. Montgomery [6], see also [8, 16, 12] and [17].

We also note that a path coalgebra is an instance of a cotensor coalgebra. Indeed consider the linearisation  $kQ_0$  of the set of vertices  $Q_0$  with its natural coalgebra structure, namely each vertex is a group-like. The vector space  $kQ_1$  is a  $kQ_0$ -bicomodule via

$$\delta_L(a) = t(a) \otimes a \quad \delta_R(a) = a \otimes s(a).$$

The cotensor product of  $kQ_1$  with itself is the kernel of  $\delta_R \otimes 1 - 1 \otimes \delta_L$ .

We denote  $Q_n$  the set of paths of length  $n$ . One immediately notice that the cotensor square of  $kQ_1$  is  $kQ_2$ , where the later is considered with its natural  $kQ_0$ -bicomodule structure. In other words, the cotensor coalgebra over the trivial coalgebra  $kQ_0$  of the bicomodule  $kQ_1$  is the path coalgebra we have described above.

**Definition 2.2** *Let  $x$  and  $y$  be vertices. The  $(y, x)$ -isotypic component of a  $kQ_0$ -bicomodule  $Z$  is*

$${}^y Z^x = \{z \in Z \mid \delta_L(z) = y \otimes z \text{ and } \delta_R(z) = z \otimes x\}.$$

*In case of  $kQ_n$  it corresponds to the vector space of  $n$ -paths from  $x$  to  $y$ .*

We record now a useful consequence, namely the universal property enjoyed by  $kQ$ . A coalgebra map  $\psi : X \rightarrow kQ$  is given by a sequence of maps  $\psi_n : X \rightarrow kQ_n$ , and is uniquely determined by two maps  $\psi_0 : X \rightarrow kQ_0$  a coalgebra map and  $\psi_1 : X \rightarrow kQ_1$  a bicomodule map, where  $X$  is to be considered as a  $kQ_0$ -bicomodule via  $\psi_0$ . For instance, the map  $\psi_2 : X \rightarrow kQ_2$  is provided by the composition

$$X \xrightarrow{\Delta} X \otimes X \xrightarrow{\psi_1 \otimes \psi_1} kQ_1 \otimes kQ_1$$

which image is indeed contained in the cotensor square  $kQ_2$ . More generally  $\psi_n = \psi_1^{\otimes n} \circ \Delta^{(n)}$  where  $\Delta^{(n)}$  denotes  $n$  successive applications of the comultiplication, a well defined map since  $\Delta$  is coassociative.

### 3 Hopf quivers

Hopf quivers are defined below, they are precisely the quivers such that the path coalgebra can be endowed with a graded Hopf algebra structure. Those

quivers are similar to Cayley graphs, which have set of vertices given by the elements of a group and arrows corresponding to multiplication by elements of a chosen system of generators.

A *ramification data*  $r$  of a group  $G$  is a positive central element of the group ring of  $G$ , i.e.  $r = \sum_{c \in C} r_C C$  is a formal sum of conjugacy classes of  $G$  with positive integer coefficients.

**Definition 3.1** Let  $G$  be a group and  $r$  be a ramification data. The corresponding Hopf quiver has set of vertices the elements of  $G$  and has  $r_C$  arrows from  $x$  to  $cx$  for each  $x \in G$  and  $c \in C$ .

A Hopf quiver is connected if and only if the union of the conjugacy classes affording non zero coefficients in the ramification data generates  $G$ .

**Example 3.2** Let  $G$  be a cyclic group which can be infinite, let  $K$  be a generator of  $G$  and  $r = K^2$ . The Hopf quiver of  $G$  is connected if  $|G|$  is finite and odd, otherwise it has two connected components. In case  $G$  is finite the connected components are crowns.

**Theorem 3.3** Let  $Q$  be a quiver. The path coalgebra  $kQ$  admits a graded Hopf algebra structure if and only if  $Q$  is the Hopf quiver of some group with respect to a ramification data.

**Proof :** If the path coalgebra  $kQ$  is a graded Hopf algebra, the set  $Q_0$  of vertices is the set of group-likes, hence  $Q_0$  is a group and the sub-coalgebra  $kQ_0$  turns into a sub-Hopf algebra, namely the group algebra of  $Q_0$ . Moreover the arrows provide a sub-vector space  $kQ_1$  which is simultaneously a  $kQ_0$ -bimodule and a  $kQ_0$ -bicomodule with structure maps compatible, since  $\delta_L$  and  $\delta_R$  are  $kQ_0$ -bimodule maps.

Such structures are called *Hopf bimodules*; we recall in the next section their classification. It turns out that the dimension of an isotypic component

$${}^y(kQ_1)^x = \{\alpha \in kQ_1 \mid \delta_L(\alpha) = y \otimes \alpha \text{ and } \delta_R(\alpha) = \alpha \otimes x\}$$

is constant when  $yx^{-1}$  remains in the same conjugacy class. In other words the number of arrows from  $x$  to  $y$  only depends on the conjugacy class of  $yx^{-1}$ . Consequently the quiver is the Hopf quiver of the ramification data which coefficients are the dimensions of the isotypic components.

Conversely, let  $Q$  be the Hopf quiver of a group  $G$  with respect to a ramification data  $r$ . We will show in the next section that there always exist a  $kG$ -Hopf bimodule  $B$  such that the dimensions of the isotypic components of  $B$  are prescribed by  $r$ , namely such that

$$\dim_k {}^y B^x = r_{\langle yx^{-1} \rangle}.$$

Actually there are several Hopf bimodules affording the same ramification data. We chose one of them and we provide the vector space  $kQ_1$  with the transported Hopf bimodule structure : chose a basis of  $B$  which respects the isotypic components and identify the vector spaces  $B$  and  $kQ_1$  using the set of arrows as a basis of  $kQ_1$ . At this point we have a group structure on the vertices of  $Q$  and a compatible multiplication of vertices by arrows. Using the quoted universal property of path coalgebras this data uniquely determines an associative coalgebra morphism and  $kQ$  becomes a pointed bialgebra. The existence of the antipode is granted by Takeuchi's results [16] p. 572 since the group-like elements of  $kQ$  are invertible.

Next we display the formula for the product of two arrows which is obtained directly from the description we made at the end of the preceding section of the map in degree 2. Let  $a$  and  $b$  be arrows of the quiver. Then

$$a.b = [t(a).b][a.s(b)] + [a.t(b)][s(a).b].$$

Note for instance that  $t(a).b$  denotes the left action of the group element  $t(a)$  on the arrow  $b$ . We record that the two homogeneous terms of  $a.b$  starts at the vertex  $s(a)s(b)$  and ends at  $t(a)t(b)$ . The above formula is a particular case of the following Theorem which describes the product of two paths.

**Definition 3.4** *A  $p$ -thin split of a path  $\alpha$  is a sequence  $\alpha_p, \alpha_{p-1}, \dots, \alpha_1$  of vertices and arrows such that the product  $\alpha_p \alpha_{p-1} \dots \alpha_1$  (in the path algebra sense) is  $\alpha$ .*

**Example 3.5** Let  $\alpha = cba$  be a length 3 path. A 7-thin cut of  $\alpha$  is for instance  $(t(c), t(c), c, b, s(b), s(b), a)$ .

Clearly  $p$ -thin splits of a  $n$ -path  $\alpha$  are in one-to-one correspondence with  $p$ -sequences of zeros and ones such that the number of ones is  $n$ . We denote  $D_n^p$  the set of such sequences. More precisely, if  $d \in D_n^p$  and  $(a_n, \dots, a_1)$  is the

sequence of arrows of  $\alpha$ , the  $p$ -thin split  $d\alpha = ((d\alpha)_p, \dots, (d\alpha)_1)$  is determined by “ $(d\alpha)_i$  is a vertex if  $d_i = 0$  and is an arrow if  $d_i = 1$ ”. The 7-thin cut of the example above corresponds to  $d = (0, 0, 1, 1, 0, 0, 1)$ .

Let  $\alpha$  be a  $n$ -path and  $\beta$  be a  $m$ -path. Let  $d \in D_n^{n+m}$  and let  $\bar{d} \in D_n^{n+m}$  be the complement sequence obtained from  $d$  by replacing each 0 by a 1 and each 1 by a 0. Consider the element

$$(\alpha.\beta)_d = [(d\alpha)_{m+n} \cdot (\bar{d}\beta)_{m+n}] \cdots [(d\alpha)_1 \cdot (\bar{d}\beta)_1]$$

which lies in the  $(m+n)$ -cotensor power of  $B$  and belongs to the isotypic component of type  $(t(\alpha)t(\beta), s(\alpha)s(\beta))$ .

**Remark 3.6** If  $d_i = 1$ , the element  $(\bar{d}\beta)_i$  is a vertex which acts on the right on the arrow  $(d\alpha)_i$  and  $[(d\alpha)_i \cdot (\bar{d}\beta)_i]$  denotes the result of the action. Conversely, if  $d_i = 0$  the above expression is the result of the left action of the group-vertex element  $(d\alpha)_i$  on the arrow  $(\bar{d}\beta)_i$ .

**Remark 3.7** The source of the first term of  $(\alpha.\beta)_d$  is

$$s[(d\alpha)_1 \cdot (\bar{d}\beta)_1] = s((d\alpha)_1)s((\bar{d}\beta)_1) = s(\alpha)s(\beta)$$

and the sequence of terms of  $(\alpha.\beta)_d$  is concatenated.

**Theorem 3.8** *Let  $G$  be a group,  $k$  be a field,  $B$  be a  $kG$ -Hopf bimodule and let  $C$  be the associated cotensor coalgebra endowed with its Hopf algebra structure (see the proof of Theorem 3.3). Chose a basis of the isotypic components of the bicomodule  $B$  and identify the coalgebra  $C$  with the path coalgebra of the quiver of  $B$ . Let  $\alpha$  and  $\beta$  be respectively paths of length  $n$  and  $m$ . Then*

$$\alpha.\beta = \sum_{d \in D_n^{n+m}} (\alpha.\beta)_d$$

**Remark 3.9**  $|D_m^{n+m}| = \binom{m+n}{n}$ .

The proof of the Theorem uses the following result :

**Lemma 3.10** Let  $\Delta_2$  be the comultiplication of the path coalgebra  $kQ \otimes kQ$ . Then

$$\Delta_2^{(p)}(\alpha \otimes \beta) = \sum_{\substack{\alpha_{(p)} \cdots \alpha_{(1)} = \alpha \\ \beta_{(p)} \cdots \beta_{(1)} = \beta}} \alpha_{(p)} \otimes \beta_{(p)} \otimes \cdots \otimes \alpha_{(1)} \otimes \beta_{(1)}$$

**Proof :** Note that the sum of the formula is over all the  $p$ -splits of  $\alpha$  and  $\beta$ . Recall that  $\Delta_2 = (1 \otimes \tau \otimes 1)(\Delta \otimes \Delta)$  where  $\tau$  is the flip map. Hence

$$\Delta_2(\alpha \otimes \beta) = \Sigma \alpha_{(2)} \otimes \beta_{(2)} \otimes \alpha_{(1)} \otimes \beta_{(1)}.$$

The result follows by induction.

**Proof of the Theorem:** we use the universal property of  $C$  with respect to  $X = C \otimes C$ . We already have in low degrees

$$\varphi_0 : C \otimes C \rightarrow kG \otimes kG \rightarrow kG \subset C$$

$$\varphi_1 : C \otimes C \rightarrow kG \otimes B + B \otimes kG \rightarrow B \subset C$$

where the first maps for  $\varphi_0$  and  $\varphi_1$  are respectively the projections to the 0 and 1 homogeneous component of  $C \otimes C$ . In order to describe  $\varphi_{n+m}(\alpha \otimes \beta)$ , we first use the Lemma for obtaining  $\Delta^{(n+m)}(\alpha \otimes \beta)$ . The morphism  $\varphi_1^{\otimes(n+m)}$  only retains pairs of complementary  $(n+m)$ -thin splits. The formula follows immediately from this remark.

**Example 3.11** If  $a$  and  $b$  are arrows, the 2-thin splits of  $a$  are  $(a, s(a))$  and  $(t(a), a)$ . The complement thin splits of  $b$  are respectively  $(t(b), b)$  and  $(b, s(b))$ . Then

$$a.b = [a.t(b)][s(a).b] + [t(a).b][a.s(b)].$$

**Example 3.12** Let  $a$  be an arrow and  $\beta = cb$  be a 2-path. The 3-thin splits of  $a$  and the corresponding complement 3-thin splits of  $\beta$  are

$$\begin{pmatrix} t(a), & t(a), & a \\ c, & b, & s(b) \end{pmatrix}$$

$$\begin{pmatrix} t(a), & a, & s(a) \\ c, & t(b), & b \end{pmatrix}$$

$$\begin{pmatrix} a, & s(a), & s(a) \\ t(c), & c, & b \end{pmatrix}$$

Then

$$\begin{aligned} a.(cb) &= [t(a).c][t(a).b][a.s(b)] + \\ &\quad [t(a).c][a.t(b)][s(a).b] + \\ &\quad [a.t(c)][s(a).c][s(a).b]. \end{aligned}$$

We record that if  $u$  is a vertex and  $\alpha = (a_n, \dots, a_1)$  is a path, then

$$\begin{aligned} u \cdot (a_n \cdots a_1) &= [u \cdot a_n] \cdots [u \cdot a_1] \\ (a_n \cdots a_1) \cdot u &= [a_n \cdot u] \cdots [a_1 \cdot u]. \end{aligned}$$

Next we will display a formula for the product of a sequence of  $n$  arrows  $A = (a_n, \dots, a_1)$ . We stress that there is no need for the arrows to be concatenated. Recall that the vertices of the quiver are group elements which acts on both sides on the vector space generated by the arrows.

Let  $\sigma$  be a permutation of the set  $\{1, \dots, n\}$ . We define a map  $p_k^\sigma$  which assigns to each arrow  $a_i$  either its source or terminus vertex, according to  $i$  being already reached by  $\sigma$  acting on  $\{1, \dots, k-1\}$ :

$$p_k^\sigma(a_i) = \begin{cases} s(a_i) & \text{if } i \notin \sigma\{1, \dots, k-1\} \\ t(a_i) & \text{if } i \in \sigma\{1, \dots, k-1\}. \end{cases}$$

Given a sequence of arrows  $A = (a_n, \dots, a_1)$ , a permutation  $\sigma$  and an integer  $k \in \{1, \dots, n\}$  we define an homogeneous element

$$A_k^\sigma = \left( p_k^\sigma(a_n) \cdots p_k^\sigma(a_{\sigma(k)+1}) \right) \cdot a_{\sigma(k)} \cdot \left( p_k^\sigma(a_{\sigma(k)-1}) \cdots p_k^\sigma(a_1) \right).$$

Note that for a fixed  $\sigma$  the elements  $A_k^\sigma$  are concatenated as  $k$  increases. Finally we set  $A^\sigma = A_n^\sigma \cdots A_1^\sigma$  where the product is the usual one in the path algebra sense.  $A^\sigma$  is an homogenous element of degree  $n$  from the vertex  $s(a_n) \cdots s(a_1)$  to the vertex  $t(a_n) \cdots t(a_1)$ .

**Proposition 3.13** *Let  $A$  be a sequence of non necessarily concatenated arrows  $(a_n, \dots, a_1)$ . Then*

$$a_n \cdots a_1 = \sum_{\sigma \in S_n} A_\sigma.$$

The proof follows from the product formula of paths that we provided above.



**Example 3.14** For  $n = 3$ , consider  $A = (c, b, a)$ . Then

$$\begin{aligned} c.b.a = & [c.tb.ta][sc.b.ta][sc.sb.a] + \\ & [tc.b.ta][c.sb.ta][sc.sb.a] + \\ & [c.tb.ta][sc.tb.a][sc.b.sa] + \\ & [tc.tb.a][tc.b.sa][c.sb.sa] + \\ & [tc.tb.a][c.tb.sa][sc.b.sa] + \\ & [tc.b.ta][tc.sb.a][c.sb.sa]. \end{aligned}$$

The first row corresponds to the identity permutation, next we use the three transpositions and finally the elements of order 3.

We provide now examples of Hopf algebras obtained through the procedure of path coalgebras.

**Example 3.15** Let  $G = \{1\}$  be the trivial group and  $B = k$  be the trivial Hopf bimodule. The ramification data is  $r = 1$ , and the quiver is a loop  $x$ . The path coalgebra corresponds to the polynomial algebra in one variable, while the path algebra structure corresponds to the usual product of polynomials. The formula above for the multiplication of paths provides the product

$$X^n . X^m = \binom{n+m}{m} X^{n+m}$$

**Example 3.16** Let  $G$  be a cyclic group,  $K$  be a generator, and consider the  $kG$  Hopf bimodule defined as follows ( $q$  is a non-zero element if  $G$  is infinite or a root of unity if  $G$  is finite) : the isotypic components  $K^{i+1} B^{K^i}$  are one-dimensional with basis  $E_i$ . Other components  $K^j B^{K^i}$  are zero. The left action is by translation

$$K.E_i = E_{i+1}$$

while the right action is by translation and multiplication by  $q$

$$E_i.K = qE_{i+1}.$$

The corresponding Hopf quiver is a crown or a quiver of type  $A_\infty^\infty$ . As for any path coalgebra a natural basis is provided by the set of paths. In this example the basis consists of concatenated sequences of arrows  $E_i$  and 0-length paths (i.e. vertices).

The Gauss binomial coefficient  $\binom{n}{i}_q$  is defined using the algebra  $k\{x, y\}/\langle yx - qxy \rangle$ , we have  $(x + y)^n = \sum_0^n \binom{n}{i}_q x^i y^{n-i}$ . The recursive formula  $\binom{n+1}{i}_q = \binom{n}{i}_q + q^{n+1-i} \binom{n}{i-1}_q$  enables to prove the equality  $\binom{n}{i}_q = \frac{n_q!}{(n-i)_q! i_q!}$  where  $n_q = 1 + q + \dots + q^{n-1}$  and  $n_q! = n_q(n-1)_q \dots 1_q$  (see for instance [10]).

**Proposition 3.17** *Let  $E_i^n$  denotes the path of length  $n$  with source vertex  $K^i$  (the index  $i$  is to be considered modulo the order of  $G$  in case  $G$  is finite). Then*

$$E_i^n \cdot E_j^m = q^{jn} \binom{n+m}{n}_q E_{i+j}^{n+m}.$$

**Proof :** We consider the product  $E_0^n \cdot E_0^m$ . Using the formula of the Theorem this product is a multiple of  $E_0^{n+m}$  by a scalar which we denote  $X_n^{n+m}$ . By induction  $X_n^{n+m} = \binom{n+m}{n}_q$  since  $X_1^{m+1} = \binom{m+1}{1}_q$  and the associativity formula

$$E_0^1 \cdot (E_0^n \cdot E_0^m) = (E_0^1 \cdot E_0^n) \cdot E_0^m$$

provides the recursive formula  $\binom{n+m+1}{1}_q X_n^{n+m} = \binom{n+1}{1}_q X_{n+1}^{n+m+1}$  which is clearly satisfied while replacing  $X_n^{n+m}$  by  $\binom{n+m}{n}_q$ . Finally we record that  $K^i \cdot E_0^n = E_i^n$  and  $E_0^n \cdot K^j = q^{nj} E_j^n$ . Hence

$$E_i^n \cdot E_j^m = K^i \cdot E_0^n \cdot K^j \cdot E_0^m = q^{nj} K^{i+j} \cdot E_0^n \cdot E_0^m.$$

## 4 Hopf bimodules

The proof of the structure Theorem of the preceding section makes use of the classification of Hopf bimodules over a group algebra obtained in [5], see also [3, 1, 7]. Let  $k$  be a field,  $G$  be a group,  $\mathcal{C}$  be the set of conjugacy classes,  $Z_C$  be the centralizer of some element in the conjugacy  $C$  and  $\text{mod } kZ_C$  be the category of right  $kZ_C$ -modules.

**Theorem 4.1** *The category of Hopf bimodules  $\mathcal{B}(kG)$  is equivalent to the cartesian product of categories  $\prod_{C \in \mathcal{C}} \text{mod } kZ_C$ .*

For a complete proof we refer to the quoted references. In order to comply with the requirements of the preceding section, we describe the functor  $W$  which

associates to a Hopf bimodule  $B$  the family  $\{u^{(C)}B^1\}_{C \in \mathcal{C}}$  where  $u(C)$  is some element in  $C$  and

$$u^{(C)}B^1 = \{b \in B \mid \delta_L(b) = u(C) \otimes b \text{ and } \delta_R(b) = b \otimes 1\}.$$

Note that  $u^{(C)}B^1$  is a right  $kZ_C$ -module. In this setting the corresponding ramification data is

$$r = \sum_{C \in \mathcal{C}} (\dim_k u^{(C)}B^1)C.$$

In order to provide at least one Hopf algebra structure to the path coalgebra of a Hopf quiver with ramification data  $r$ , one can chose a family  $\{M_C\}_{C \in \mathcal{C}}$  of trivial  $kZ_C$ -modules of adapted dimension, namely  $\dim_k M_C = r_C$ . More generally, the following result is now evident :

**Theorem 4.2** *Let  $G$  be a group,  $r$  be a ramification data and  $Q$  be the corresponding Hopf quiver. The complete list of graded Hopf algebra structures on the path coalgebra  $kQ$  is in one to one correspondence with the set of collections  $\{M_C\}_{C \in \mathcal{C}}$  where  $M_C$  is a right  $kZ_{u(C)}$ -module of dimension  $r_C$ .*

**Example 4.3** Consider  $G$  a cyclic group (finite or not) with a generator  $K$ , and the ramification data  $r = K$ . In other words, only one conjugacy class is highlighted by  $r$  and have coefficient one. Hence we consider the set of one-dimensional  $kG$ -modules, in order to obtain the complete list of Hopf algebras structures on the path coalgebra of the corresponding Hopf quiver.

A one-dimensional  $kG$ -module is a non-zero scalar  $q$  if  $G$  is infinite and a  $|G|$ -root of unity  $q$  if  $G$  is finite. The resulting Hopf bimodule is obtained through the functor  $V$  of [3], and is the Hopf bimodule we have described in the last example of the preceding section.

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